

IMSC 2058 Solution for Homework 5

We first show that (i) and (ii) are equivalent, i.e. A mapping $f : X \rightarrow Y$ between topological spaces X and Y is said to be continuous if and only if for any open subset V in Y , its inverse image under f , $f^{-1}(V)$, is an open subset of X .

Suppose that f is continuous, Let V be open in Y . To show $f^{-1}(V)$ is open it suffices to show that each point in $f^{-1}(V)$ has an open neighborhood that is contained in $f^{-1}(V)$. Let $x \in f^{-1}(V)$. By the continuity of f at x , there is an open neighborhood of x that is mapped into V and therefore is contained in $f^{-1}(V)$. Conversely, if f^{-1} maps open sets to open sets, then it is immediate that f is continuous on all of X .

Then we show (ii) and (iii) are equivalent.

(ii) \Rightarrow (iii)

Let F be a closed set in Y . Then $Y \setminus F$ is open in Y . Note that

$$f^{-1}(Y \setminus F) = X \setminus (f^{-1}(F))$$

it follows from (ii) that $X \setminus (f^{-1}(F))$ is open. Hence $f^{-1}(F)$ is closed.

(iii) \Rightarrow (ii)

Suppose that V is an open set in Y . Then $Y \setminus V$ and $f^{-1}(Y \setminus V)$ are closed. Since

$$f^{-1}(Y \setminus V) = X \setminus (f^{-1}(V))$$

which implies that $f^{-1}(V)$ is open.

Therefore, the three statements are equivalent.